POINT AND LINE

First, we will introduce you with basic formulas and their applications:

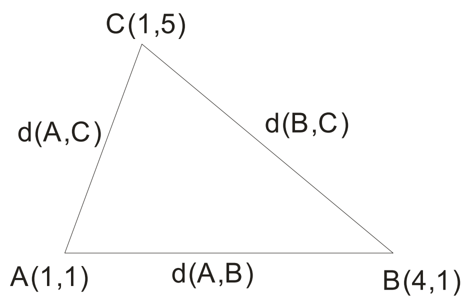
1. The distance between two points

If they give us points $A(x_1, y_1)$ i $B(x_2, y_2)$, then the distance between them is:

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1.

Determine the length of the triangle page if we have *A(1,1) , B(4,1) i C(1,5)*



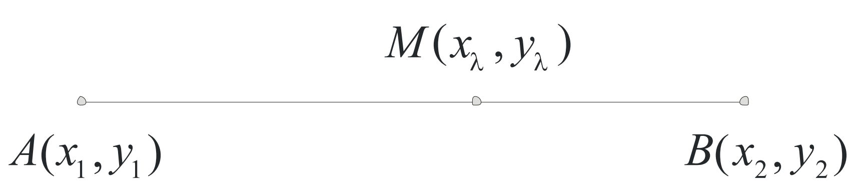
One note: it does not matter whether you mark d (A, B) or d (B, A) because the solution is the same!

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d(A,B) = \sqrt{(4-1)^2 + (1-1)^2} = \sqrt{9+0} = 3$$
$$d(A,C) = \sqrt{(1-1)^2 + (5-1)^2} = \sqrt{0+16} = 4$$
$$d(B,C) = \sqrt{(1-4)^2 + (5-1)^2} = \sqrt{9+16} = 5$$

2. Long division in a given relation

If the point $M(x_\lambda, y_\lambda)$ is internal point of long-AB, where are $A(x_1, y_1)$ and $B(x_2, y_2)$ and $AM : MB = \lambda$ $(\frac{AM}{MB} = \lambda)$, then:

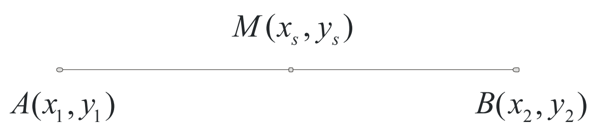
$$M(x_\lambda, y_\lambda) \rightarrow x_\lambda = \frac{x_1 + \lambda x_2}{1+\lambda} \quad \text{and} \quad y_\lambda = \frac{y_1 + \lambda y_2}{1+\lambda}$$



3. How to find the middle of a longer AB

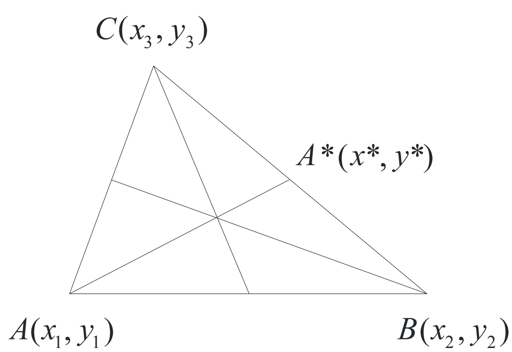
If the point $M(x_s, y_s)$ is the middle of AB ($A(x_1, y_1)$ and $B(x_2, y_2)$) , then:

$$M(x_s, y_s) \to x_s = \frac{x_1 + x_2}{2} \quad \text{i} \quad y_s = \frac{y_1 + y_2}{2}$$



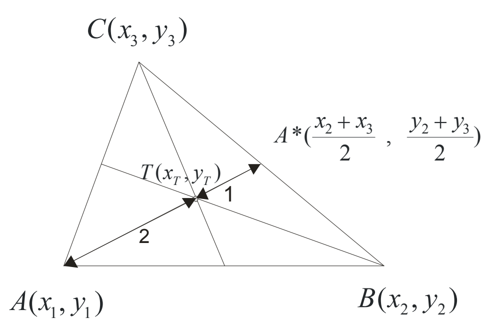
Example 2.

focus of the triangle



First, we find the coordinates of point $A^*(x^*, y^*)$, middle of BC.

$$A^*(x^*, y^*) \to x^* = \frac{x_2 + x_3}{2} \quad \text{and} \quad y^* = \frac{y_2 + y_3}{2}$$



Know that AT : TA* = 2 : 1 = 2

$$T(x_T, y_T) \to x_T = \frac{x_1 + 2(\frac{x_2 + x_3}{2})}{1 + 2} = \frac{x_1 + x_2 + x_3}{3} \quad \text{and} \quad y_T = \frac{y_1 + 2(\frac{y_2 + y_3}{2})}{1 + 2} = \frac{y_1 + y_2 + y_3}{3}$$

4. Area of a triangle using the coordinates of vertices

$A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC , then:

$$A_{\triangle} = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

or through determinants:

$$A_{\triangle} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Absolute value is there to provide us that a solution is not negative, because the area can not be a negative number.

Example 3.

Calculated area of a triangle ABC if A(-2,3) ; B(8,-2) and C(3,8)

$$A_{\triangle} = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$
$$A_{\triangle} = \frac{1}{2}\left|-2(-2-8) + 8(8-3) + 3(3-(-2))\right|$$
$$A_{\triangle} = \frac{1}{2}\left|-2(-10) + 8 \cdot 5 + 3(3+2)\right|$$
$$A_{\triangle} = \frac{1}{2}\left|20 + 40 + 15\right|$$
$$A_{\triangle} = \frac{1}{2}\left|75\right|$$
$$A_{\triangle} = 37,5$$

LINE

i) **general (implicit form) is** $ax+by+c=0$

ii) the **explicit form is** $y=kx+n$

This form is our most important because it is used in many formulas.

k- direction ($k=tg\alpha$,where α is the angle that the line build with a positive direction of x - axis)

n - a segment on the y- axis

How to move from the general in explicit form?

$$ax+by+c=0$$
$$by=-ax-c$$
$$y=-\frac{a}{b}x-\frac{c}{b}$$
$$k=-\frac{a}{b} \quad \text{and} \quad n=-\frac{c}{b}$$

Example 4.

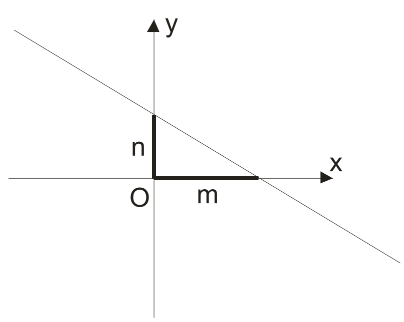
7x+3y + 23=0 transfer in explicit form and find k and n

$$7x+3y+23=0$$
$$3y=-7x-23$$
$$y=-\frac{7}{3}x-\frac{23}{3}$$
$$k=-\frac{7}{3} \quad \text{and} \quad n=-\frac{23}{3}$$

iii) $\frac{x}{m}+\frac{y}{n}=1$ **the segment form**

m – a segment on the x- axis

n – a segment on the y- axis



Example 5.

In equation $px+(p+1)y-8=0$ *find* p *,* so that the segment of x- axis is two times higher than the segment of y-. axis.

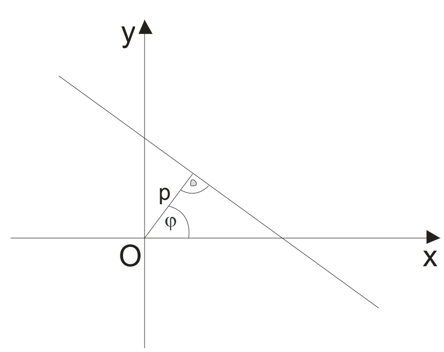
From the text of the task we can conclude that is $m=2n$

$$px+(p+1)y-8=0$$
$$px+(p+1)y=8$$
$$\frac{px}{8}+\frac{(p+1)y}{8}=1$$
$$\frac{x}{\frac{8}{p}}+\frac{y}{\frac{8}{p+1}}=1 \rightarrow m=\frac{8}{p} \quad \text{and} \quad n=\frac{8}{p+1}$$

Substituting in $m=2n$, we have:

$$m=2n$$
$$\frac{8}{p}=2\cdot\frac{8}{p+1}$$
$$\frac{8}{p}=\frac{16}{p+1}$$
$$16p=8(p+1)$$
$$16p=8p+8$$
$$16p-8p=8$$
$$8p=8$$
$$p=1$$

iv) $x\cos\varphi+y\sin\varphi=p$ **the normal form**



In this equation is:

p is the normal distance from (0,0) to our line

φ is the angle that the distance p build with positive direction of x- axis

The formula for the transition from the general to the normal form is:

$$ax+by+c=0 \rightarrow \frac{ax+by+c}{\pm\sqrt{a^2+b^2}}=0$$

but careful, take sign in front of root as sign opposite the sign of c.

Example 6.

Reduce the equation 4x - 3y +5 = 0 to normal form

$4x-3y+5=0 \rightarrow \frac{4x-3y+5}{-\sqrt{4^2+3^2}}=0$ (- infront root because c=5)

$\frac{4x-3y+5}{-\sqrt{25}}=0 \rightarrow \frac{4x-3y+5}{-5}=0 \rightarrow -\frac{4}{5}x+\frac{3}{5}y-1=0$ from here we have

p=1, $\cos\varphi=-\frac{4}{5}$, $\sin\varphi=\frac{3}{5}$

v) **Line through the point** $A(x_1,y_1)$ with the direction k is: $y-y_1=k(x-x_1)$

vi) **Line through the point** $A(x_1,y_1)$ **and** $B(x_2,y_2)$ is : $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

Notes that $k=\frac{y_2-y_1}{x_2-x_1}$

What can be reciprocal position of two lines in the plane?

1) **Can be cut**

Intersection point found by solving the system of these two equations!

If we have $y=k_1x+n_1$ and $y=k_2x+n_2$ **then the angle of cut** is given by the formula:

$$tg\alpha=\left|\frac{k_2-k_1}{1+k_1\cdot k_2}\right|$$

If the two lines cut at right angles, $k_1\cdot k_2=-1$ **(normality condition)**

2) **can be parallel**

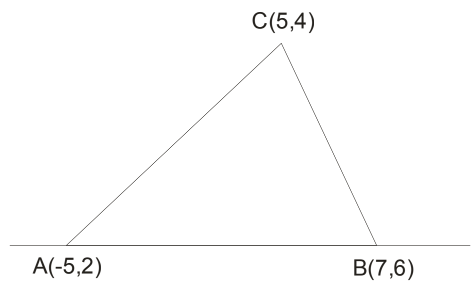
$y=k_1x+n_1$ and $y=k_2x+n_2$ are **parallel if** $k_1=k_2$ **(Condition parallels)**

Example 7.

The vertices of the triangle are A(-5,-2), B(7,6), C(5,4). Find:

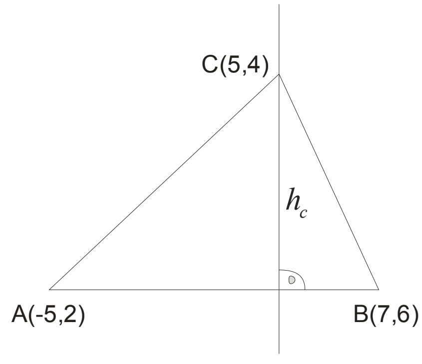
a) *equation of line AB*
b) *equation of* h_c
c) *the angle at the vertices A*

a) Use formula: **Line through the point** $A(x_1, y_1)$ **and** $B(x_2, y_2)$ is : $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$



$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - (-2) = \frac{6 - (-2)}{7 - (-5)}(x - (-5))$$
$$y + 2 = \frac{6 + 2}{7 + 5}(x + 5)$$
$$y + 2 = \frac{8}{12}(x + 5)$$
$$y + 2 = \frac{2}{3}(x + 5)$$
$$y = \frac{2}{3}x + \frac{2}{3} \cdot 5 - 2$$
$$y = \frac{2}{3}x + \frac{10}{3} - \frac{6}{2}$$
$$y = \frac{2}{3}x + \frac{4}{3}$$

b)



The equation of line h_c we find as the equation of line through one point C (5,4) and his direction must satisfies the condition normality with the line AB

Direction of line AB : $y=\frac{2}{3}x+\frac{4}{3}$ is $k_1=\frac{2}{3}$.

$$k_2=-\frac{1}{k_1} \qquad y-y_1=k(x-x_1)$$

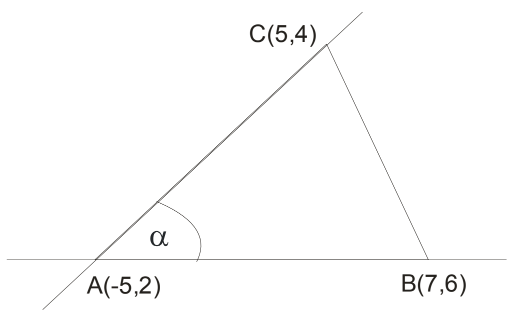
$$k_2=-\frac{1}{\frac{2}{3}} \qquad y-4=-\frac{3}{2}(x-5)$$

$$y=-\frac{3}{2}x+\frac{15}{2}+4$$

$$k_2=-\frac{3}{2} \qquad y=-\frac{3}{2}x+\frac{23}{2}$$

c) The angle at the vertices of A is the angle between the line AB and AC.

$$tg\alpha=\left|\frac{k_2-k_1}{1+k_1\cdot k_2}\right|$$



From line AB we have direction $k_1=\frac{2}{3}$.

Direction of AC we will find:

A(-5,-2), C(5,4) change in $k=\frac{y_2-y_1}{x_2-x_1}$

$$k_2=\frac{y_2-y_1}{x_2-x_1}$$

$$k_2=\frac{4-(-2)}{5-(-5)}$$

$$k_2=\frac{6}{10}$$

$$k_2=\frac{3}{5}$$

$$tg\alpha = \left|\frac{k_2 - k_1}{1 + k_1 \cdot k_2}\right| \to tg\alpha = \left|\frac{\frac{3}{5} - \frac{2}{3}}{1 + \frac{3}{5} \cdot \frac{2}{3}}\right| \to tg\alpha = \left|\frac{-\frac{1}{15}}{1 + \frac{6}{15}}\right| \to tg\alpha = \left|\frac{-\frac{1}{15}}{\frac{21}{15}}\right|$$

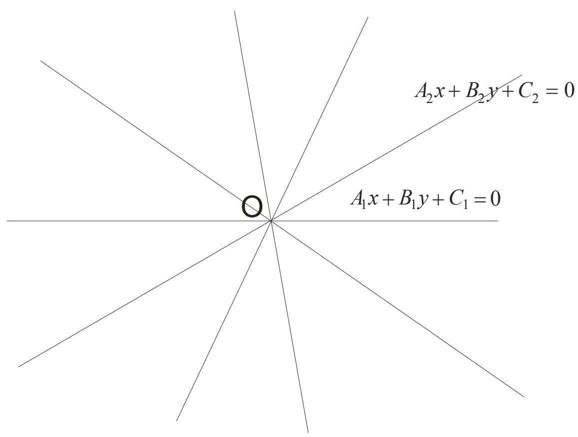
$$tg\alpha = \frac{1}{21}$$

$$\alpha = arctg\frac{1}{21}$$

strand of lines

If we have $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ two line equations that intersect at the point O, then:

$$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0$$



So, to describe the **strand of lines we need two lines!**

Distance $A(x_1, y_1)$ from the line $ax + by + c = 0$ is given by formula:

$$d = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$

Example 8.

In $2x+y+4+\lambda(x-2y-3)=0$ *find line whose distance from point P(2,-3)is* $\sqrt{10}$.

$$2x+y+4+\lambda(x-2y-3)=0$$
$$2x+y+4+\lambda x-2\lambda y-3\lambda=0$$
$$(2+\lambda)x+(1-2\lambda)y+4-3\lambda=0$$

From here , we have $a=2+\lambda,\quad b=1-2\lambda$

$$d=\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$$

$$d=\frac{|(2+\lambda)\cdot 2+(1-2\lambda)\cdot(-3)+4-3\lambda|}{\sqrt{(2+\lambda)^2+(1-2\lambda)^2}}$$
$$\sqrt{10}=\frac{|4+2\lambda-3+6\lambda+4-3\lambda|}{\sqrt{4+4\lambda+\lambda^2+1-4\lambda+4\lambda^2}}$$
$$\sqrt{10}=\frac{|5\lambda+5|}{\sqrt{5\lambda^2+5}}$$

From here we have two solutions: $\lambda_1=1$, $\lambda_2=-\dfrac{9}{10}$

Substituting this in $2x+y+4+\lambda(x-2y-3)=0$ and we have:

3x – y + 1 = 0

11x + 28 y +67 = 0